Chapter - 8

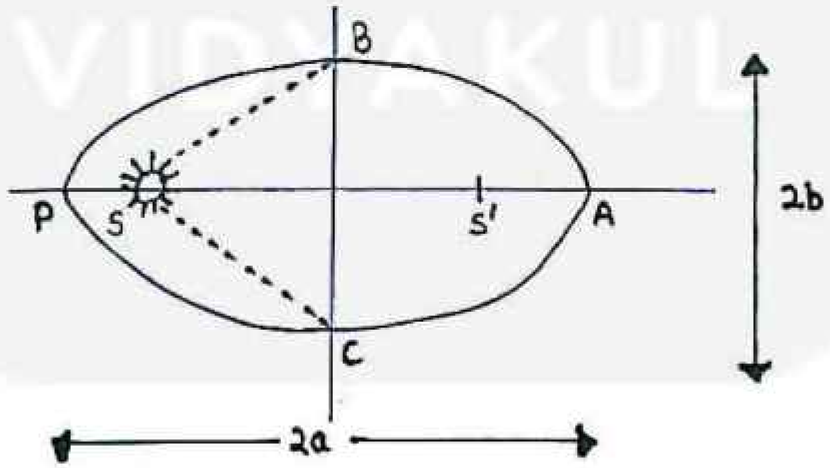
Gravitation

Kepler's Laws

Johannes kepler formulated three Laws which describe planetary motion. They are as follows:

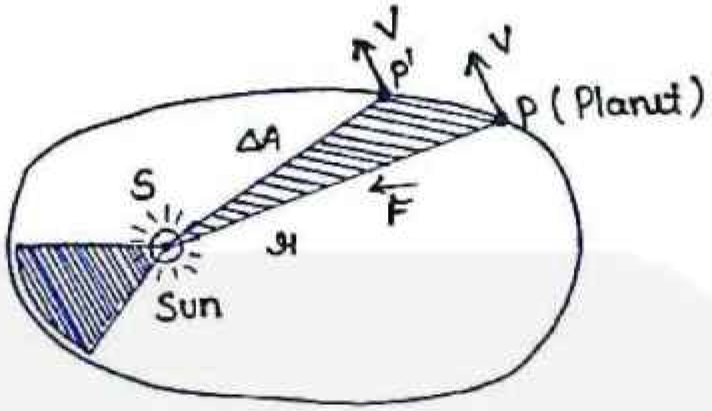
(i) **Law of Orbits** –

Each planet revolves around the Sun in an elliptical Orbit with the Sun at one of the foci of the ellipse.



(ii) **Law of areas** –

The speed of planet varies in such a way that the radius, vector drawn from the Sun to planet sweeps out equal areas in equal times.



(iii) Law of periods –

The square of the time period of revolution is proportional to the cube of the semi-major axis of the elliptical orbit.

i.e.

$$\boxed{T^2 \propto r^3}$$

If r_1 and r_2 are the shortest and the longest distances of the planet from the Sun, the semi-major axis is given by –

$$\left(\frac{r_1 + r_2}{2}\right)$$

Newton's Law of Gravitation

Newton's Law of gravitation states that every particle in the universe attracts every other particle with a force directly proportional to the product of their masses and inversely proportional to the square of the distance between them. The direction of the force is along the line joining the particles.

$$F \propto m_1 m_2$$

and

$$F \propto \frac{1}{r^2}$$

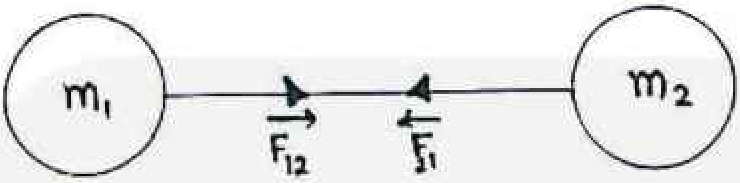
so

$$\boxed{F = G\frac{m_1 m_2}{r^2}}$$

where G is a constant, called as the Universal Constant of gravitation.

⇒ Vectorially the gravitation force is given as

$$\boxed{\vec{F} = \frac{G m_1 m_2}{r^2}\hat{r}}$$



$$\boxed{\vec{F}_{12} + \vec{F}_{21} = 0}$$

The gravitational force between two particles form an action - reaction pair.

⇒ The value of 'G' has been experimentally determined. i.e. $G = 6.67 \times 10^{-11}\ Nm^2 kg^{-2}$.

⇒ Dimensional formula of G is $[M^{-1}L^3T^{-2}]$.

Universal Constant of gravitation G is numerically equal to the force of attraction between two particles of unit mass each separated by unit distance.

⇒ Important Characteristics of Gravitational force

(i) Gravitational force between two bodies is a Central force i.e. it acts along the line joining the Centres of the two interacting bodies.

(ii) Gravitational force between two bodies is independent of the nature of the intervening medium.

(iii) Gravitational force between two bodies does not depend upon the presence of Other bodies.

(iv) It is valid for point objects and Spherically Symmetrical Objects.

(v) Magnitude of force is extremely small.

Acceleration due to Gravity of the Earth -

The acceleration produced in a body on account of the force of gravity is known as acceleration due to gravity. It is usually denoted by 'g'. It is always towards the Centre of Earth.

Let us suppose you are Standing on the top of the building with a small stone in your hand. Mass of the stone be 'm'. when you throw the Stone on the ground, the gravitational force of the earth attracts the stones downwards. The gravitational force acting on the stone is $F = mg$.

Also, the force between two objects is given by the universal Law of gravitation. So here one object is the stone and object is the earth.

$$\boxed{F = \frac{GMm}{d^2}}$$

where M = mass of the earth

m = mass of the stone

d = distance

$$\Rightarrow mg = \frac{GMm}{d^2}, \text{ or}$$

$$g = \frac{GM}{d^2}$$

Suppose the object is on the surface of the earth or nearby.

Now, In this case,

$$d = R + h$$

So,

$$\boxed{g = \frac{GM}{(R+h)^2}}$$

Let us calculate the value of g on the earth. i.e. $h = 0$.

$$\boxed{g = \frac{GM}{R^2}}$$

$G = 6.7 \times 10^{-11} \ Nm^2/kg^2$

$M = 6 \times 10^{24} \ kg$

$R = 6.4 \times 10^{6} \ m$

So, putting the Values —

$$g = \frac{6.7 \times 10^{11} \times 6 \times 10^{24}}{(6.4 \times 10^{6})^{2}}$$

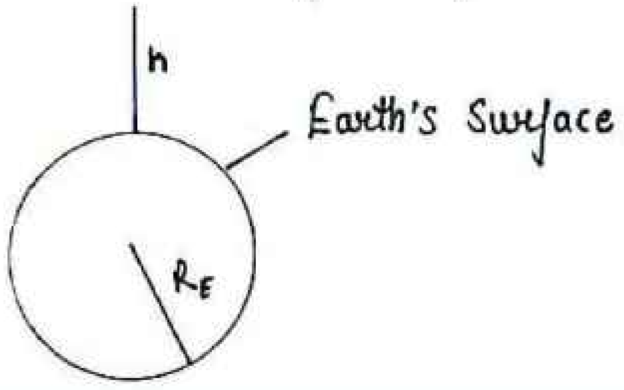
$$\boxed{g = 9.8 \ m/s^{2}}$$

This is the value of acceleration due to gravity. The value of this acceleration due to gravity changes from place to place. It is not universal Constant.

Acceleration Due to Gravity Below and above the Surface of the earth —

(i) Above the surface of the earth

Let us consider a point mass m at a height h above the Surface of the earth.



Assume the radius of the earth be R_e and the mass of earth be M_e.

Now, as we know that the mass of earth is concentrated at its Centre. The magnitude of force on the point mass m will be

$$F_h = \frac{G M_e m}{(R_e + h)^2} \quad \text{———} \quad (i)$$

The acceleration experienced by that point mass will be

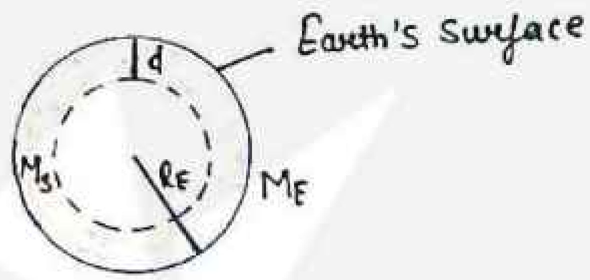
$$g_h = \frac{F}{m} = \frac{G M_e m}{(R_e + h)^2 m}$$

$$\boxed{g_h = \frac{G M_e}{(R_e + h)^2}}$$

from above equation, move above the surface of the earth the value of g reduces.

(ii) Below the surface of the Earth -

Consider a mass m at A which is present at depth d below the surface of the earth. Consider the radius of earth as R_e and a mass of M_e.



As per our discussion, the gravitational force on point mass m will be due to the smaller sphere of radius $(R_e - d)$ whose mass M_s is concentrated at the centre.

Now,

$$M_s = \frac{4}{3}\pi (R_e - d)^3 \rho$$

$$M_e = \frac{4}{3}\pi R_e^3 \rho$$

$$\boxed{\frac{M_s}{M_e} = \left[\frac{R_e - d}{R_e}\right]^3}$$

By universal Law of gravitation

$$\boxed{F_d = GM_e m \frac{(R_e - d)}{R_e^3}}$$

from Newton's second Law that

$$g_d = \frac{F}{m}$$

$$g_d = GM_e \frac{R_e - d}{R^3}$$

$$g_d = g\frac{R_e - d}{R_e} \quad \cdots \quad \left[\therefore g = \frac{GM_e}{R_e^2}\right]$$

$$g_d = g\left[1 - \frac{d}{R_e}\right]$$

Thus, as we go down below the earth's surface, the acceleration due to gravity decreases.

Mass and Mean Density of Earth -

Mass and Mean density of Earth is

given -

Mass of earth, $\boxed{M = \frac{gR^2}{G}}$

Mean density of earth

$$\boxed{\rho = \frac{3g}{4\pi GR}}$$

Gravitational Potential energy

Gravitational Potential is the work done per unit mass to bring that body from infinity to that point. It is represented by V. SI unit of gravitational potential is J/kg. It is the potential body arising out of the force of gravity. If due to the force, if the position changes, then the change in the potential energy is the work done on the body by the force.

Case- 1- 'g' is constant

Assume an object at point A and later it moves to point B. So, in this case, the work done is force X displacement.

$$\boxed{W_{BA} = \text{Force} \times \text{displacement}}$$

force is nothing but the gravitational force exerted by the earth. The height of point A from the surface of the earth is h_2 and that of point B is h_1.

$$W_{BA} = mg(h_2 - h_1)$$

$$= mgh_2 - mgh_1$$

the work done in moving the object is the difference in its potential energy between its final and initial position.

Case - 2- 'g' is not Constant

Let the position vector of the first object be r_1 and the position vector of the second object be r_2. So here the work done in bringing the object from one position to another is -

$$W = \int_{r_1}^{r_2} F\,dr$$

where, $\boxed{F = \frac{GM_e m}{r^2}}$

$$W = \int_{r_1}^{r_2} GM_e m\,dr$$

$$= -GM_e m\left[\frac{1}{r_2} - \frac{1}{r_1}\right]$$

Escape Speed

The minimum velocity required to project a body vertically upward from the surface of earth so that it comes out of the gravitational field of earth is

Called escape Speed.

Suppose the ball is initially in your hand. So that is the initial position of the ball. Now throw the ball at a greater Velocity that it never comes back. As we don't know where did the ball go, so its final Velocity is ∞.

At initial position,

Total energy = kinetic energy + Potential energy

or,

$$T.E. = K.E. + P.E.$$

$$\boxed{\text{Kinetic energy} = \frac{1}{2} m v^2}$$

$$\boxed{\text{Potential energy} = \frac{-GMm}{R_e + h}}$$

Here, M is the mass of the earth and m is the mass of the ball. At the earth's surface,

$$P.E.(0) = 0$$

Hence,

$$T.E.(0) = \frac{1}{2} mv^2$$

At final position (∞)

$$K.E. = \frac{1}{2} mv_f^2$$

$$P.E. = \frac{-GM_e m}{R_e + h} = 0 \quad \{h = \infty\}$$

Now by the Law of Conservation of energy, the total energy at the initial position should be equal to the final position.

$$T.E.(\infty) = T.E.(0)$$

or $$\frac{1}{2} mv_f^2 = \frac{1}{2} mv_i^2 - \frac{GM_e m}{R_e + h}$$

L.H.S. has to be always +ve, which implies,

$$\frac{1}{2} m v_i^2 - \frac{G M_e m}{R_e + h} \geq 0$$

$$\Rightarrow \quad \frac{1}{2} m v_i^2 = \frac{G M_e m}{R_e + h}$$

$$\Rightarrow \quad \boxed{v_i^2 = \frac{2 G M_e}{R_e + h}}$$

Assume the ball is thrown from earth's surface.

$$h << R_e \Rightarrow R_e + h \sim R_e$$

$$\Rightarrow \quad v_i^2 = \frac{2 G M_e}{R_e}.$$

$$\Rightarrow \quad \boxed{v_i = \sqrt{\frac{2 G M_e}{R_e}}}$$

This is the velocity in which the objects never comes back.

In terms of 'g'

$$g = \frac{GM}{R^2}$$

$$\Rightarrow \quad gR_e = \frac{GM_e}{R_e}$$

$$\boxed{V_e = \sqrt{2gR_e}}$$

Earth Satellites

A Satellite is a body which is revolving Continuously in an Orbit around a Comparatively much larger body.

When the satellite is Orbiting around the planet is because of two reasons. The first reason is that there is a gravitational force between the satellite and the planet. The second reason is that it just wants to speed past the planet. It just wants to go out of the Orbit. Satellites are two types.

1) Natural Satellites

2) Artifical Satellites

Time period of Earth Satellites

Time taken by the Satellite to Complete one rotation around the earth. Suppose a Satellite keeps on revolving around them in a Circular Orbit. So as it moves in Circular motion, there is a Centripetal force acting on it.

$$F_c = \frac{mv^2}{r^2}$$

$$F_c = \frac{mv^2}{R_e + h}$$

'h' is the distance above earth's Surface. This Centripetal force will act towards the Centre. Now there is another gravitatio-nal force between the earth and the

Satellite that is,

$$F_G = \frac{GmM_e}{(R_e+h)^2}$$

Now, $F_c = F_G$, implies

$$\frac{mv^2}{R_e+h} = \frac{GmM_e}{(R_e+h)^2}$$

$$\Rightarrow \quad v^2 = \frac{-GM_e}{R_e+h}$$

$$\Rightarrow \quad \boxed{v = \sqrt{\frac{GM_e}{R_e+h}} = \text{Velocity}}$$

Calculate the time period of the satellite.

Satellite covers a distance of $2\pi(R_e+h)$ in one revolution.

$$T = \frac{\text{distance}}{\text{Velocity}} = \frac{2\pi(R_e+h)}{v}$$

$$\boxed{T = \frac{2\pi (R_e + h)^{3/2}}{\sqrt{GM_e}}}$$

Thus this is the time period taken by the Satellite to revolve around the earth.

Energy of Orbiting Satellite

We know that m is the mass of the Satellite and the Velocity with which it moves is v. So, it given by $\frac{1}{2} mv^2$

$$\boxed{v = \sqrt{\frac{GM_e m}{R_e + h}}}$$

So, the kinetic energy $\frac{1}{2} \frac{GM_e m}{R_e + h}$

Now, the potential energy $\frac{-GM_e m}{R_e + h}$

Total energy = kinetic energy + Potential energy

$$= \frac{1}{2}\frac{GM_e m}{R_e+h} + -\frac{GM_e m}{R_e+h}$$

$$\boxed{\text{Total energy} = -\frac{GM_e m}{2(R_e+h)}}$$

Physical Quantity	Symbol	Dimensions	Unit	Remarks
Gravitational Constant	G	$[M^{-1}L^{3}T^{-2}]$	$Nm^2 kg^{-2}$	6.67×10^{-11}
Gravitational Potential Energy	$V(r)$	$[ML^{2}T^{-2}]$	J	$-\frac{GMm}{r}$ (Scalar)
Gravitational Potential	$u(r)$	$[L^{2}T^{-2}]$	$J kg^{-1}$	$-\frac{GM}{r}$ (Scalar)
Gravitational Intensity	E or g	$[LT^{-2}]$	ms^{-2}	$\frac{GM}{r^2}\hat{r}$ (Vector)