Chapter -7

Systems of particles and Rotational Motion

A rigid body is a body with a perfectly definite and Unchanging Shape. The distances between all pairs of particles of Such a body do not Change.

Centre of Mass

For a System of Particles, the Centre of Mass is defined as that point where the entire mass of the system is imagined to be Concentrated, for consideration of its translational motion.

If all the external forces acting on the body/System of bodies were to be applied at the centre of Mass, the state of rest/motion of the body/System of bodies Shall remain unaffected.

rotational motion here is fixed. So there is no need to change the angular Velocity. Angular accelaration is $\alpha = \frac{d\omega}{dt}$. So

the kinematics equations of linear motion with uniform accelaration is,

$$\boxed{V = V_0 + at}$$

$$\boxed{X = X_0 + V_0 t + \frac{1}{2} at^2}$$

$$\boxed{V^2 = V_0^2 + 2ax}$$

where X_0 = initial displacement

V_0 = initial Velocity

here initial means $t = 0$.

Now, this equation corresponds to the kinematics equation of the rotational motion.

are analogue to each Other. In rotational motion, the angular Velocity is ω which is analogous to the linear Velocity v in the transitional motion.

Let us discuss further the kinematics of rotational motion about a fixed point. The kinematic quantities in rotational motion like the angular displacement θ, angular velocity ω and angular acceleration α respectivly corresponds to kinematic quantities in linear motion like displace-ment x, velocity v and accelaration a.

Angular Velocity

Angular Velocity is the time rate of Change of angular displacement.

$$\boxed{\omega = \frac{d\theta}{dt}}$$

the same as that determined by the actual distribution of mass of the body is called radius of gyration.

If we consider that the whole mass of the body is concentrated at a distance K from the axis of rotation, then moment of inertia, I can be expressed as

$$\boxed{I = MK^2}$$

Where M is the total Mass of the body and K is the radius of gyration.

$$K = \sqrt{\frac{r_1^2 + r_2^2 + \cdots + r_n^2}{n}}$$

Kinematics of Rotational Motion about a fixed point –

Rotational motion and translational motion their respective perpendicular distance from the axis.

$$I = m_1 r_1^2 + m_2 r_2^2 + --- + m_n r_n^2$$

$$= \sum_{i=1}^{n} m_i r_i^2$$

where m_i is the mass and r_i the distance of the i^{th} particle of the rigid body from the axis of rotation.

It is measured in kg m^2 and has the dimension of $[ML^2]$.

Radius of Gyration

The distance of a point in a body from the axis of rotation, at which if whole of the mass of the body were supposed to be Concentrated, its moment of inertia about the axis of rotation would be made the balance possible.

The reaction of your fingertip at the Center is equal and opposite to Mg. This balance is an example of translational and rotational equilibrium.

The Center of gravity of the book is located at the point where the total torque due to force mg is zero. Center of gravity is, therefore, that point where the total gravitational torque on the rigid body is zero.

Moment of inertia

The rotational Inertia of a rigid body is referred to as its moment of inertia. The moment of inertia of a body about an axis is defined as the sum of the products of the masses of the particles Constituting the body and the Square of

translational equilibrium. Rotational equilibrium is attained when $d_1F_1 - d_2F_2 = 0$.
For rotational equilibrium, the sum of moments about the fulcrum is zero.

Here, F_1 = load. F_2 = effort needed to lift the load, d_1 = load arm and d_2 = effort arm.

Centre of Gravity

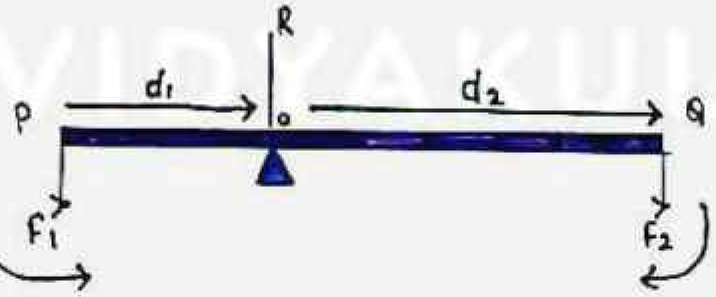
Center of gravity is the point of balance of a rigid body. This situation is the result of the mechanical equilibrium between the two rigid bodies

Example - When you hold a book on the tip of your finger the center at which the book is balanced is called the Centre of gravity. The mechanical equilibrium between the finger and book has

Principle of Moments

for Understanding mechanical equilibrium we need to understand the working of a fulcrum and lever. Those pose as the best examples of mechanical equilibrium.

A See-Saw in parks explains the principle of lever and fulcrum. See-Saw is the lever while the point at which the rod is pivoted is the fulcrum.



The lever here shows mechanical equilibrium. R, the reaction of the support from the fulcrum, is directed opposite the forces, F_1 and F_2 and at $R - F_1 - F_2 = 0$, we see that the rigid body attains zero then the body shows translational equilibrium as the linear momentum remains unchanged despite the change in time.

When we sum up the above finding -

$$F_1 + F_2 + F_3 + F_4 + --- + F_n = F_i = 0$$

(for translational equilibrium)

$$\tau_1 + \tau_2 + \tau_3 + \tau_4 + --- + \tau_n = \tau_i = 0$$

(for rotational equilibrium)

These equations are the vector in nature. As scalars the forces and torque in their x, y and z components are :

$F_{ix} = 0$, $F_{iy} = 0$ and

$F_{iz} = 0$

$\tau_{ix} = 0$, $\tau_{iy} = 0$ and

$\tau_{iz} = 0$

remains Conserved. Mathematically, If

$$\vec{\tau}_{ext} = \vec{0}$$

$$\vec{L} = \vec{L_1} + \vec{L_2} + --- + \vec{L_n} = a\,(\text{Constant})$$

Equilibrium of a rigid body

Equilibrium is a defined as any point where the total amount of external force or torque is zero. This point may be anywhere near the Center of Mass. External force in translational motion of the rigid body Changes the linear momentum of that body.

While the external torque in rotational motion can Change the angular momentum of the rigid body.

⇒ If the total force on a rigid body is

the momentum and the perpendicular distance.

It is given by

$$\boxed{\vec{L} = \vec{r} \times \vec{p}}$$

SI Unit of angular momentum is "$kg\,m^2 s^{-1}$" and its dimensional formula is $[M^1 L^2 T^{-1}]$

Geometrically, the angular momentum of a particle is equal to twice the product of its mass and the area velocity. i.e,

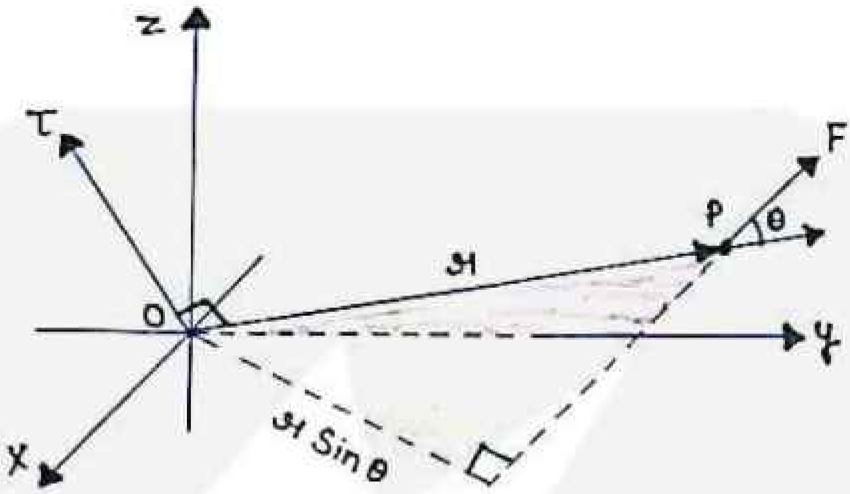
$$L = 2m \times \frac{dA}{dt}$$

⇒ Torque (τ) and angular momentum are correlated as:

$$\boxed{\vec{\tau} = \frac{d\vec{L}}{dt}}$$

⇒ If no net external torque acts on a system

It is measured in Nm and has dimensions of $[ML^2T^{-2}]$.



Angular Momentum of particle

The Angular momentum (or moment of momentum) about an axis of rotation is a vector quantity, whose magnitude is equal to the product of the magnitude of momentum and the perpendicular distance of the line of action of momentum from the axis of rotation and its direction is perpendicular to the plane containing

The unit of angular acceleration is rad s^{-2} and dimensional formula is $[M^0 T^{-2} L^0]$.

Torque and Angular Momentum

Torque is the moment of force. Torque acting on a particle is defined as the product of the magnitude of the force acting on the particle and the perpendicular distance of the application of force from the axis of rotation of the particle.

Torque or moment of force	= Force X perpendicular distance

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

$$= rF \sin\theta \, \hat{n}$$

Where θ is smaller angle between $\vec{r}$ and $\vec{F}$; $\hat{n}$ is unit vector along $\vec{r}$.

The direction of angular Velocity is along the axis of rotation. It is measured in radian / Sec and its dimensional formula is $[M^0 L^0 T^{-1}]$.

The relation between angular Velocity and linear Velocity is given by

$$\boxed{\vec{v} = \vec{\omega} \times \vec{r}}$$

Angular Acceleration

The Angular acceleration of a body is defined as the change in the angular Velocity to the time interval.

$$\boxed{\text{Angular acceleration} = \frac{\text{Change in Angular Velocity}}{\text{Time taken}}}$$

$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$

Moreover $\hat{j} \times \hat{i} = -\hat{k}$

$\hat{k} \times \hat{j} = -\hat{i}$

$\hat{i} \times \hat{k} = -\hat{j}$

(g) In terms of Components

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Angular Velocity and its Relation with Linear Velocity -

The Angular Velocity of a body or a particle is defined as the angular displacement of the body or the particle to the time interval during which this displacement occurs.

$$\boxed{\omega = \frac{d\theta}{dt}}$$

(a) for parallel as well as anti - parallel Vectors (i.e. when $\theta = 0^\circ$ or 180°), the Cross - product is zero.

(b) Magnitude of cross - product of two perpendicular vectors is equal to the product of the magnitudes of the given vectors.

(c) Vector product is anti - commulative.

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

(d) Vector product is distributive.

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

(e) $\vec{A} \times \vec{B}$ does not change sign under reflection

$$(-\vec{A}) \times (-\vec{B}) = \vec{A} \times \vec{B}$$

(f) for unit orthogonal vectors, We have

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

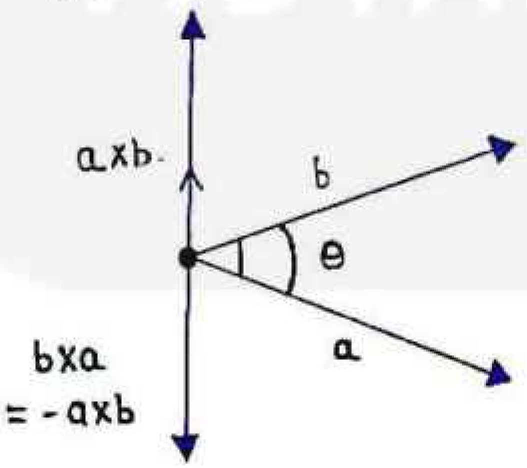
$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}$$

Sine of the smaller angle between them.

If θ is the smaller angle between $\vec{A}$ and $\vec{B}$, then

$$\vec{A} \times \vec{B} = \vec{C} = AB \sin\theta \, \hat{C}$$

Where $\hat{C}$ is a unit vector in the direction of $\vec{C}$. the direction of $\vec{C}$ or $\hat{C}$ (i.e. vector product of two vectors) is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$ and pointing in the direction of advance of a right handed screw when rotated from $\vec{A}$ to $\vec{B}$.



⇒ Some important properties of Cross-product are as follow -

Second Law to a System of particles. If the total external force acting on the System is zero.

$$\boxed{F_{ext} = 0} \text{ then, } \boxed{\frac{dP}{dt} = 0}$$

This means that P = Constant. So whenever the total force acting on the system of a particle is equal to zero then the total linear momentum of the System is Constant or Conserved. This is nothing but the Law of Conservation of total linear momentum of a System of particles.

Vector Product or Cross Product of two Vectors

The vector product or cross product of two vectors $\vec{A}$ and $\vec{B}$ is another $\vec{C}$, whose magnitude is equal to the product of the magnitudes of the two vectors and

$$mV = \sum m_i v_i$$

So, comparing these equation we get,

$$\boxed{P = MV}$$

Therefore, we can say that the total linear momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its center of mass.

Differentiating the above equation. we get,

$$\frac{dP}{dt} = M\frac{dV}{dt} = MA$$

$\frac{dv}{dt}$ is accelaration of centre of mass,

MA is the force external. so,

$$\frac{dP}{dt} = F_{ext}$$

This above equation is nothing but Newton's

Linear Momentum of System of particles

We know that the linear momemtum of the particle is

$$\boxed{P = mV}$$

Newton's second Law for a single particle is given by,

$$\boxed{F = \frac{dP}{dt}}$$

Where F is the force of the particle. For 'n' no. of particles total linear momentum is,

$$P = P_1 + P_2 + P_3 + --- + P_n$$

each of momentum is written as $m_1V_1 + m_2V_2 + --- + m_nV_n$.

Velocity of the centre of Mass is

$$V = \Sigma \frac{m_iV_i}{M}$$

Motion of Centre of Mass

The Centre of mass of a System of particles moves as if the entire mass of the system were concentrated at the Centre of mass and all the external forces were applied at that point. Velocity of Centre of mass of a system of two particles, m_1 and m_2 with Velocity v_1 and v_2 is given by

$$\boxed{V_{cm} = \frac{m_1 \overline{v_1} + m_2 \overline{v_2}}{m_1 + m_2}}$$

Acceleration of Centre of mass, a_{cm} of a two body System is given by

$$a_{cm} = \frac{m_1 \overrightarrow{a_1} + m_2 \overrightarrow{a_2}}{m_1 + m_2}$$

$$Y = \frac{m_1 y_1 + m_2 y_2 + \cdots + m_n y_n}{m_1 + m_2 + \cdots + m_n}$$

$$= \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}$$

$$\boxed{= \frac{1}{M} \sum_{i=1}^{n} m_i y_i}$$

$$Z = \frac{m_1 z_1 + m_2 z_2 + \cdots + m_n z_n}{m_1 + m_2 + \cdots + m_n}$$

$$= \frac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i}$$

$$\boxed{= \frac{1}{M} \sum_{i=1}^{n} m_i z_i}$$

$$\vec{r}_{cm} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + \dots + m_n\vec{r_n}}{m_1 + m_2 + \dots + m_n}$$

$$= \frac{\sum_{i=1}^{n} m_i\vec{r_i}}{\sum_{i=1}^{n} m_i}$$

⇒ The co-ordinate of the centre of mass of an n-particle system is given as:

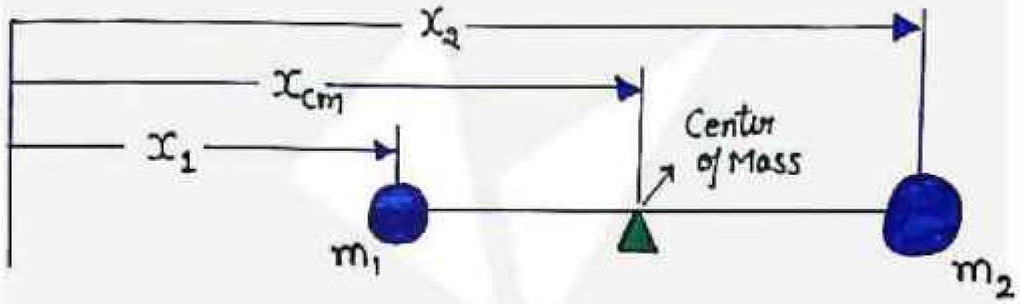
$$X = \frac{m_1\vec{r_1} + m_2\vec{r_2} + \dots + m_n\vec{r_n}}{m_1 + m_2 + \dots + m_n}$$

$$= \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

$$\boxed{= \frac{1}{M}\sum_{i=1}^{n} m_i x_i}$$

Where $\sum_{i=1}^{n} m_i = M$, mass of System

The Centre of Mass of a body or a System is its balancing point. The Centre of mass of a two-particle system always lies on the line joining the two particles and is somewhere in between the particles.



For two masses

$$\boxed{x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}}$$

For a system of n particle of masses $m_1, m_2, m_3 \text{---} m_n$ and their respective position vectors $\vec{r_1}, \vec{r_2} \text{----} \vec{r_n}$;

the position

Law of Conservation of Angular Momentum

According to the Law of Conservation of angular momentum, if there is no external couple acting, the total angular momentum of a rigid body or a system of particles is conserved.

If the moment of inertia of the body changes from I_1 and I_2 due to the change of the distribution of mass of the body, then angular velocity of the body changes from $\vec{\omega_1}$ and $\vec{\omega_2}$,

$$I_1 \vec{\omega_1} = I_2 \vec{\omega_2}$$

$$\boxed{I_1 \omega_1 = I_2 \omega_2}$$

L_ℓ and L_z as the Components of L that are perpendicular to the z-axis and along the z-axis respectively.

$$L_\ell = OB_t \times m_t V_t$$

Here, m_t and V_t are mass and velocity of a t^{th} particle and B_t is the center of the circle of motion described by the particle t.

$$L_z = l_{tz} = \sum_t m_t r_t^2 \omega k'$$

$$\boxed{L_z = I \omega k'}$$

according to definition of moment of inertia $I = m_t r_t^2$

In rotational motion we take angular momentum as the sum of individual angular momentums of various particles.

$$\boxed{L = L_z + L_\ell}$$

Likewise, We can say that OB X v is perpe-ndicular to the fixed axis. Denoting a part of l along fixed axis z as l_z,

$$l_z = BA \times mv = m r_i^2 \omega$$

$$l = l_z + OB \times mv$$

l_z is parallel to the fixed axis while l is perpendicular.

Generally, angular momentum l is not along the axis of rotation which means that for any particle l and ω are not impliedly parallel to one another, but for any particle p and v are parallel to each other. for a system of particles, total angular momentum

$$L = l_t = l_{tz} + OB_t \times m_t v_t$$

$$l = (OB + BA) \times p = (OB \times p) + (BA \times p)$$

Since, $p = mv$

hence $l = (OB \times mv) + (BA \times mv)$.

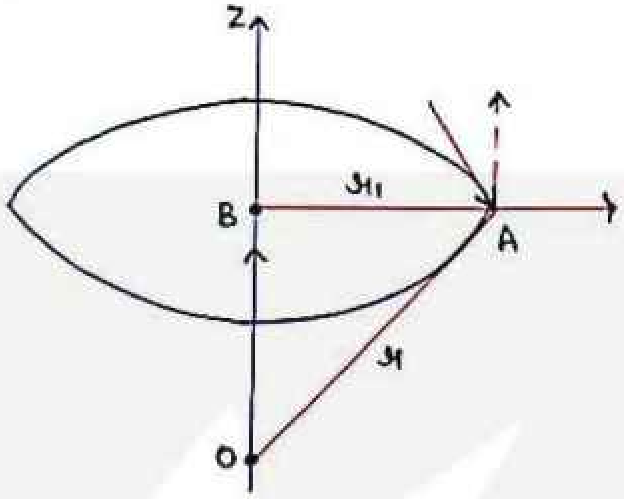
The linear Velocity (v) of the particle at point A is given by

$$\boxed{v = \omega r_1}$$

r_1 is the length of BA which is the perpendicular distance of point A from the axis of rotation. v is tangential at A to the Circular motion in which the particle moves. with the help of the right hand rule, $BA \times v$, which is parallel to the fixed axis. The unit vector along the fixed axis at k'.

$$BA \times mv = r_1 \times (mv)\, k' = m r_1^2 \omega$$

is that of a rigid body rotating about a fixed axis.



for total angular momentum (L)

$$\boxed{L = \sum_{i=1}^{N} r_i \times p_i}$$

for any particle, $l = r \times p$.

$$r = OA$$

using the right angle rule,

$$OA = OB + BA$$

Substituting these values in r we get,

Here, $d\theta$ is the angular displacement of the particle and is equal of $\angle P_1 Q P'_1$.

The workdone by the force on particle

$$= dW_1$$

$$= F_1 \, dS_1 \cos\phi_1$$

$$\boxed{= F_1 (r_1 \, d\theta) \sin\alpha_1}$$

ϕ_1 is the angle between the tangent at P_1 and the force F_1 and α_1 is the angle between radius vector and F_1.

Angular Momentum in case of Rotation about a fixed Axis -

The angular momentum of any particle rotating about a fixed axis depends on the external torque acting on that body. The angular momentum discussed here, work done by a torque acting on a particle that is rotating about a fixed axis. The particle is moving in a circular path with center Q on the axis. $P_1 P_1'$ is the arc of displacement ds_1. The graph in the figure shows the rotational motion of a rigid body across a fixed axis.

F_1 is the same force acting on particle P_1 and lies in a plane perpendicular to the axis.

This plane can be called as 'x-y' plane and r_1 is the radius of the circular path followed by particle P_1.

Now, from the figure, $QP_1 = r_1$ and particle P_1 moves to position P' in time Δt.

The displacement of the particle here is ds_1.

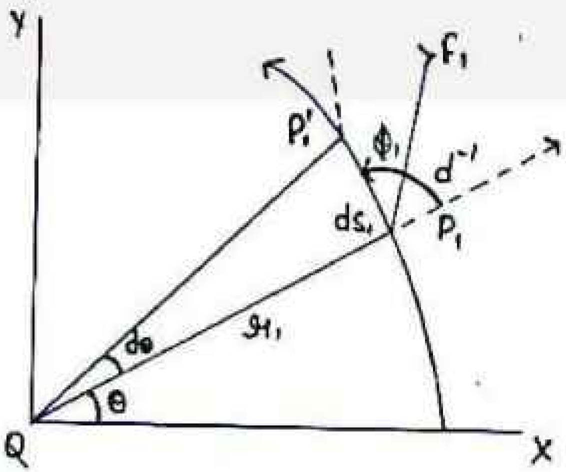
The mangnitude of $ds_1 = r_1 d\theta$

translational motion, all the particles move and behave in a similar manner. But in rotational motion, the rigid body dynamics indicate a different behaviour.

A particle in rotational motion moves with an angular velocity. Moment of inertia and torque for the rotational motion are like mass and force in translational motion.

Graphical Representation of a rigid body Dynamics –

For understanding the dynamics of a rigid body during rotational motion motion around a fixed axis.



Kinematics equations for Rotational Motion with Uniform Angular Acceleration –

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

Where θ_0 = initial angular displacement of the rotating body.

ω_0 = initial angular velocity of the particle of the body.

Dynamics of Rotational Motion about a fixed axis –

When in Motion, a rigid body is believed to be a System of particles. Each of its particles follows a path depending on the kind of motion it follows, In a