Chapter -6

Work, Energy and Power

Scalar Product

Scalar product of dot product -: This product of two vectors is a scalar quantity therefore, it is called scalar product' and it is denoted by symbol of dot (.). therefore it is also called 'Dot product'. It is give by $\vec{A} \cdot \vec{B} = AB \cos\theta$, where θ is the angle between two vectors. Example of dot product is work i.e. $W = \vec{F} \cdot \vec{d}$

Vector product or cross-product -: This product of vectors is a vector quantity, therefore, it is called 'vector product' and it is denoted by symbol of cross (x). hence it is also called 'cross product.' It is obtained by $\vec{A} \times \vec{B} = AB \sin\theta\, \hat{n}$, where n is unit vector at right angles to plane of 2 and P direction of which is decided by right hand screw rule. Example of cross product is torque, $\vec{\tau} = \vec{r} \times \vec{F}$

Notions of work and kinetic energy the work Energy Theorem

According to work-energy theorem, the work done by a force on a body is equal to the change in kinetic energy of the body.

$$w = \int F\,dx = \int ma\,dx$$

$$= m\int_{v} \frac{dv}{dt}\cdot dx$$

$$w = m\int^{v} v\,dv$$

$$\boxed{W = \frac{1}{2}mv^2 - \frac{1}{2}mu^2}$$

Work

Work is said to be done when a

force applied on the body displaces the body through a certain distance in the direction of applied force. It is measured by the product of the force and the distance moved in the direction of the force.

i.e., $W = F \cdot S$

If an object undergoes a displacement 'S' along a straight line while acted on a force F that makes an angle θ with S as shown.

The work done W by the agent is the product of the component of force in the direction of displacement and the magnitude of displacement.

i.e. $$W = FS \cos\theta = \vec{F}.\vec{S}$$

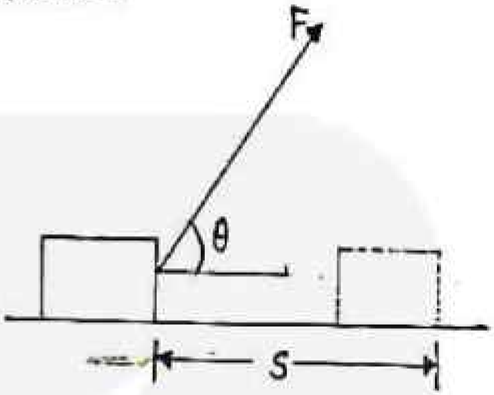
Work done is a scalar quantity measured in newtonmetre.

Its dimension is $[ML^2T^{-2}]$

(1 newton - metre = 1 joule)

Following are some significant points about work done, derived form the definition given above.

(i) Work done by a force is zero if displacement is perpendicular to the force ($\theta = 90°$).



(ii) Work done by the force is positive if angle between force and displacement is acute ($\theta < 90°$).

(iii) Work done by the force is negative if angle between force and displacement is obtuse ($\theta > 90°$).

If the applied force varies with time/position, the work done is given by:

$$\boxed{W = \int \vec{F} \cdot d\vec{s}}$$

Kinetic Energy

The energy possessed by a body by

virtue of its motion is known as its kinetic energy.
For an object of mass m and having a velocity v, the kinetic energy is given by: K.E. or $K = 1/2\ mv^2$

Work Done By a Variable Force

Graphical Method

A constant force is rare. It is the variable force which is encountered more commonly.

To evaluate the work done by a variable force, let us consider a force acting along a fixed direction, say x-axis, but having a variable magnitude.

We have to compute work in moving the body from A to B under the action of this variable force.

To facilitate this, we assume that

the entire displacement from A to B is made up of a large number of infinitesimal displacements.

One such displacement shown in the following figure from P to Q.

Since the displacement $PQ = dx$ is infinitsimally small, we consider that all along this displacement, force is constant in magnitude as well in the same direction.

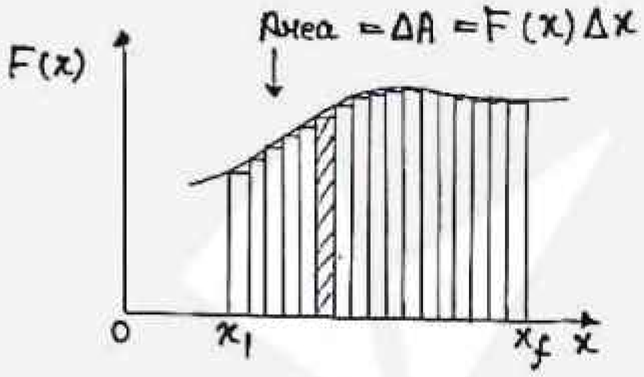
Now, a small amount of work done in moving the body from P to Q is give by,

$dW = F \times dx = (PS)(PQ) =$ area of strip PQRS

Therefore, the total work done in moving the body from A to B is given by

$$W = \sum dw$$
$$W = \sum F \times dx$$

Here, when the displacement is allowed to approach zero, then the number of terms in the sum increases without a limit. And the sum approaches a definite value equal to the area under the curve C D.



Thus, we may rewrite that

$$W = \lim_{dx \to x} \sum F\,(dx)$$

Using integral calculus, we may write it as

$$W = \int_{X_A}^{X_B} A\,(dx)$$

where

$$x_A = O_A \text{ and } x_B = OB$$

$$W = \int_{X_A}^{X_B} \text{area of strip } PQRS$$

Which is nothing but the total area

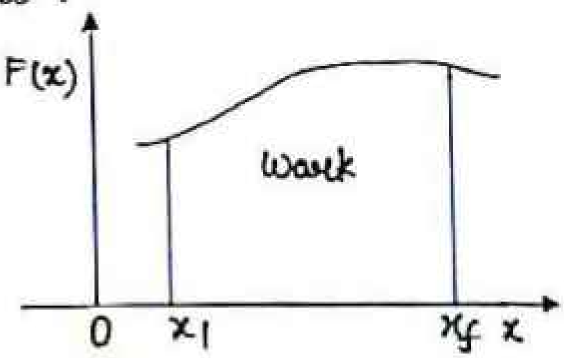
under the curve between F and x-axis from $x = x_A$ to $x = x_B$.

$$\boxed{W = \text{Area of } ABCDA}$$

Clearly the work done by a variable force is numerically equal to the area under the force curve and the displacement axis.

Mathematical Treatment (of work done by a variable force)

Suppose we have to evaluate the work done in moving a body from a point A (S_A) to point B (S_B) under the action of a varying force as shown in the following figure. Here, S_A and S_B are the distance of the points A and B with respect to some reference point.



At any stage, let the body be at P, where force on the body is $\vec{F}$.

Under the action of this force, let the body undergo an infinitesimally small displacement $d\vec{s}$

During such a small displacement, if we assume that the force remains constant, then small amount of work done in moving the body from P to Q is

given by,

$$dW = \vec{F} \cdot d\vec{s}$$

Now, when the displacement is zero, the total work done in moving the body from A to B can be obtained by integrating the above expression between S_A and S_B as follows.

$$\boxed{W = \int_{S_A}^{S_B} \vec{F} \cdot d\vec{s}}$$

Potential Energy

The energy possessed by a body by virtue of its position or condition is known as its potential energy.

There are two common forms of potential energy : gravitational and elastic.

Gravitational potential energy of a body is the energy possessed by the body by virtue of its position above the surface of the earth.
It is given by
(U) P. E. = mgh
where m ⟶ mass of a body
g ⟶ acceleration due to gravity on the surface of earth. h ⟶ height through which the body is raised.

⟶ When an elastic body is displaced from its equilibrium position, work is needed to be done against the restoring elastic force.

The work done is stored up in the body in the form of its elastic potential energy.

If an elastic spring is stretched (or compressed) by a distance Y from its equilibrium position, then its elastic potential energy is given by.

$$U = 1/2 \ Kx^2$$

where, K $\longrightarrow$ force constant of given spring

Mechanical Energy and ITS Conservation

The mechanical energy (E) of a body refers to the sum of kinetic energy (K) and potential energy (V) of the body

i.e. $$E = K + V$$

Obviously, mechanical energy of a body is a scalar quantity measured in joules.

We can show that the total mechanical energy of a system is conserved if the force doing work on the system is conservative.

This is known as the principle of conservation of total mechanical energy.

For simplicity, we assume the motion to be one dimensional only. Suppose a body undergoes a small displacement 'x' under the action of a conservative force F. According to the work energy theorem, change in kinetic energy is equal to the work done.

$$\Delta K = F(x)\Delta x$$

Now, as the force is conservative, the potential energy function V (x) is defined as

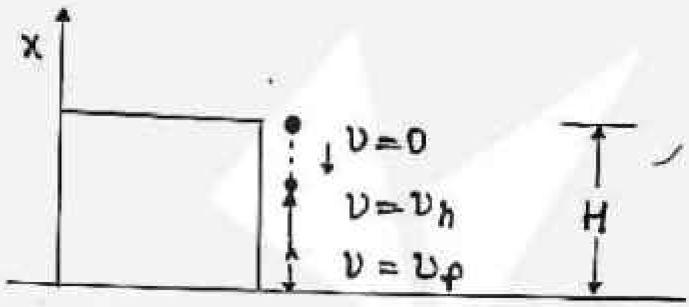
$$-\Delta V = F(x)\Delta x \text{ or } \Delta V = -F(x)\Delta x$$

Adding both the above expressions, we get,

$$\Delta K + \Delta V = 0 \text{ or } \Delta (K+V) = 0.$$

which means that

$$(K+V) = F = \text{constant}$$



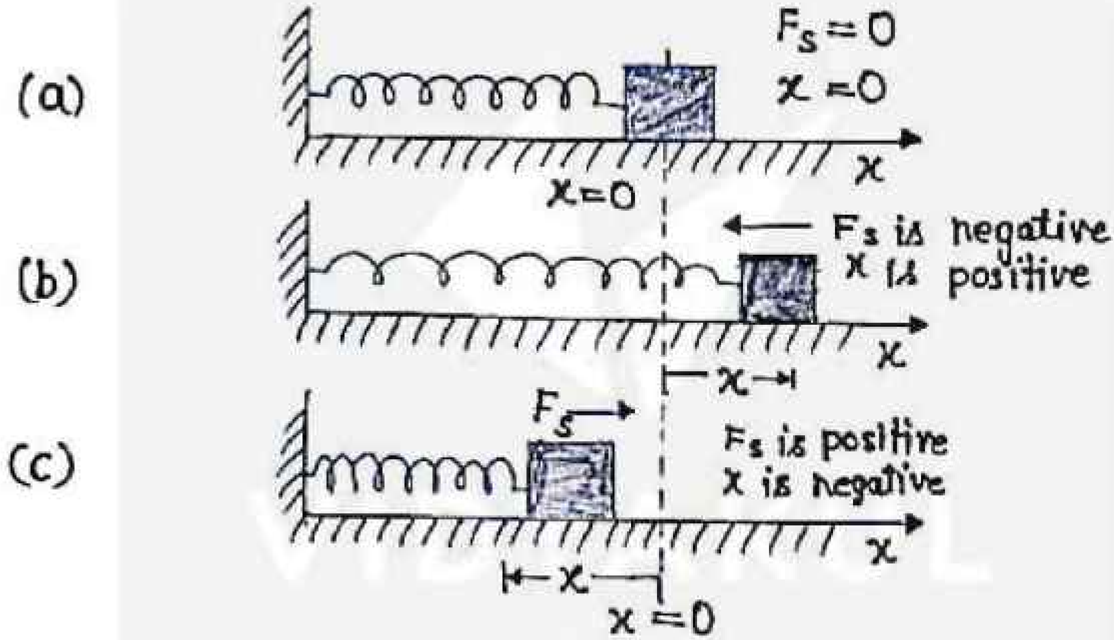
The conversion of potential energy to kinetic energy for a ball of mass m dropped from a height H.

Potential Energy of a Spring

Potential energy of a spring refers to the energy associated with the state of compression or expansion of an elastic spring.

To compute it, consider an elastic

spring OA of negligible mass. The end O of the spring is fixed to a rigid support and a body of mass 'm' lie on a perfectly frictionless horizontal table



The position of the body A, when spring is unstretched, is chose as the origin. Now, when the spring is compressed or elongated, it tends to recover to its original length, on account of elasticity. The force trying to bring the spring back to its original configuration is termed restoring force or spring force.

For a small stretch or compression, spring obeys Hook's law, i.e., for a spring.

Restoring Force $\propto$ stretch or compression

$$-F \propto x \text{ or } -F = kx$$

where k is a constant of the spring called the spring constant.
It is established that for a spring, $k \propto \frac{1}{l}$. i.e., smaller the length of the spring, greater would be the force constant and vice-versa.

The negative sign in the equation indicates that the restoring force is always directed towards the equilibrium position.

Now, consider that the body be displaced further through an infinitesimally small distance dx, against the restoring force.

A small amount of work done in increasing the length of the spring by dx is given by.

$$dW = -F\,dx = kx\,dx$$

Thus, the total work done in giving displacement x to the body can be obtained by integrating from $x=0$ to $x=x$, i.e.

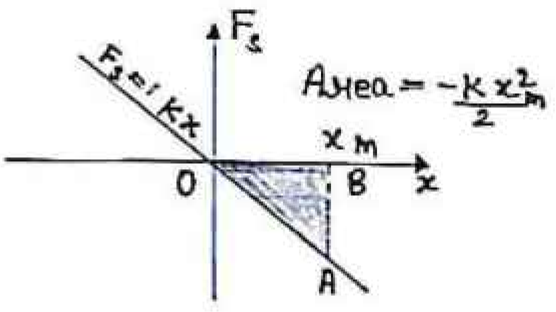
$$W = \int_{x=0}^{x=x} kx\,dx = \frac{1}{2}kx^2$$

This work done is stored in the spring at the point B.

$$PE \text{ at } B = W = \frac{1}{2}kx^2$$

The variation of potential energy with distance x is as shown in the following figure.

(d)



Power

It is the rate of doing work. i.e., the work done per unit time.

$$p = \frac{dw}{dt} = \vec{F}.\vec{v} = Fv \cos \alpha$$

where α is the angle between the force F and the velocity v.

Power is a scalar quantity. Its SI unit is watt. 1 watt $= 1\ Js^{-1}$. The dimensional formula of power is $[M^1L^2 T^{-3}]$

Other commonly used units of power are:

$$1 \text{ kilowatt} = 1\ KW = 10^3\ W$$
$$1 \text{ megawatt} = 1\ MW = 10^3\ KW = 10^6\ W$$
$$1 \text{ horse power (hp)} = 764 \text{ watt} = 0.746\ KW.$$

Collision

Collision is defined as an isolated event in which two or more colliding bodies exert relatively strong forces on each other for a relatively short time.
Collision between particles have been divided broadly into types.

(i) Elastic collisions (ii) Inelastic collisions

Elastic Collisions

A collision between two particles or bodies to be elastic if both the linear momentum and the kinetic energy of the system remain conserved.

Example: Collisions between atomic particles, atoms, marble balls and billiard balls.

Inelastic Collision

A collision is said to be inelastic if the linear momentum of the system remains conserved but its kinetic energy is not conserved.

Example: When we drop a ball of wet putty on to the floor then the collision between ball and floor is an inelastic collision.

Collision is said to be one dimensional, if the colliding particles, move along the same straight line path both before as well as after the collision.

In one dimensional elastic collision, the relative velocity of approach before collision is equal to the relative velocity of separation after collision.

If two particles of mass m_1 and m_2 moving with velocities $\vec{u}_1$ and $\vec{u}_2$ respectively collide head on such that $\vec{v}_1$ and $\vec{v}_2$ be their respective velocities after collision, then,

$$\boxed{\vec{v}_1 = \frac{(m_1-m_2)\vec{u}_1+2m_2\vec{u}_2}{(m_1+m_2)} \text{ and } \vec{v}_2 = \frac{2m_1\vec{u}_1+(m_2-m_1)\vec{u}_2}{(m_1+m_2)}}$$

Coefficient of Restitution or Coefficient of Resilience

Coefficient of restitution is defined as the ratio of relative velocity of separation after collision to the relative velocity of approach before collision.

It is represented by 'e'.

$$\boxed{e = \frac{v_2 - v_1}{u_1 - u_2}}$$

Elastic and Inelastic Collisions in Two Dimensions

Let us consider two bodies A and B moving with their initial velocities $\vec{u}_1$ and $\vec{u}_2$ respectively is a two dimensional plane. If they collide with each other and still moving with certain velocities $\vec{v}_1$ and $\vec{v}_2$ respectively after the collision, then the collision is known as two dimensional (or an oblique) collision Fig.-(i).

When the collision is elastic the total kinetic energy of the two bodies before the collision is equal to the total energy of the bodies after the collision. It means, the kinetic energy is conserved is case of elastic collision.

The kinetic energy is not conseved in case of inelastic collision.

When the body A is moving with a velocity of $\vec{u}_1$ and the body B is at rest i.e. $\vec{u}_2 = 0$, then Fig. (ii) after the collision, let θ be the angle known as scattering angle made by the body with its initial direction Fig. (iii) and the body B moves with an angle of ϕ with its initial direction. This angle is known as 'angle of recoil'.

According to the law of conservation of momentum

$$m_1 u_1 + 0 = m_1 v_1 \cos\theta + m_2 v_2 \cos\phi.$$

(i)

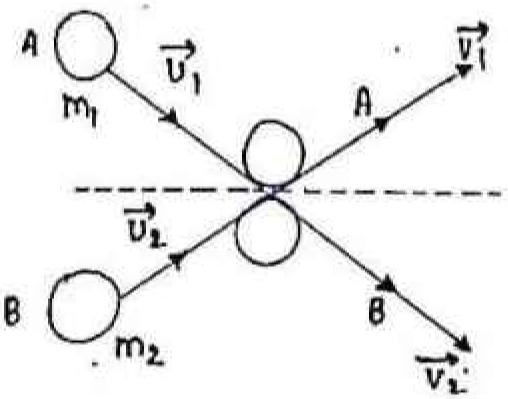


Fig (i)

and $$0 = m_1 v_1 \sin\theta - m_2 v_2 \sin\phi$$

As the collision is perfectly elastic, Total K.E. before the collision = Total K.E. after the collision.

$$\frac{1}{2} m_1 u_1^2 + 0 = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 \qquad \text{(iii)}$$

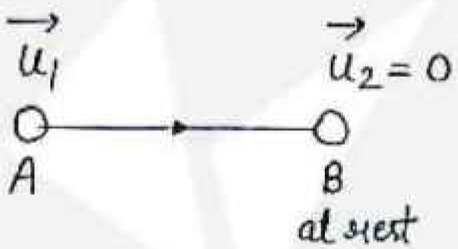


Fig (ii)

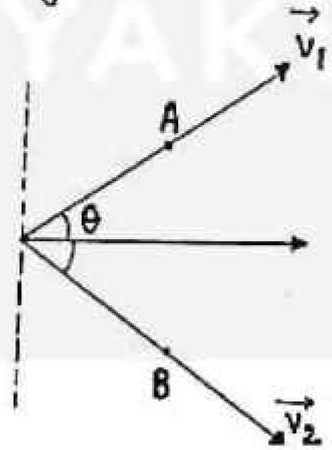


Fig (iii)

Special Case : If $m_1 = m_2$ then the above equation (i), (ii) and (iii) are reduced to

$$u_1 = v_1 \cos\theta + v_2 \cos\phi \quad (iv)$$

$$0 = v_1 \sin\theta - v_2 \sin\phi \quad (v)$$

and $$u_1^2 = V_1^2 + V_2^2 \quad (vi)$$

From eq. (iv) and (vi) we get.

$$(V_1 \cos\theta + V_2 \cos\phi)^2 = V_1^2 + V_2^2$$

$$V_1^2 \cos^2\theta + V_2^2 \cos^2\phi + 2V_1 V_2 \cos\theta \cos\phi = V_1^2 + V_2^2$$

$$2V_1 V_2 \cos\theta \cos\phi = V_1^2 - V_1^2 \cos^2\theta + V_2^2 - V_2^2 \cos^2\phi$$

$$2V_1 V_2 \cos\theta \cos\phi = V_1^2 (1-\cos^2\theta) + V_2^2 (1-\cos^2\phi)$$

$$2V_1 V_2 \cos\theta \cos\phi = V_1^2 \sin^2\theta + V_1^2 \sin^2\phi \quad (vii)$$

$$2V_1 V_2 \cos\theta \cos\phi = V_1^2 \sin^2\theta + V_1^2 \sin^2\theta$$

$$2V_1 V_2 \cos\theta \cos\phi = 2V_1^2 \sin^2\theta$$

$$V_2 \cos\theta \cos\phi = V_1 \sin^2\theta$$

$$\cos\theta = \left(\frac{V_1}{V_2}\right)\left(\frac{\sin^2\theta}{\cos\phi}\right) \qquad \text{(viii)}$$

Now $\cos(\theta+\phi) = \cos\theta\,\cos\phi - \sin\theta\,\sin\phi$

$$= \left(\frac{V_1}{V_2}\right)\left(\frac{\sin^2\theta}{\cos\phi}\right)\cdot\cos\phi - \sin\theta\left(\frac{V_1}{V_2}\right)$$

$\sin\theta$ [From (v)]

$$= \frac{V_1}{V}\sin^2\theta - \frac{V_1}{V_2}\sin^2\theta$$

$$= 0$$

$$\boxed{\theta + \phi = \frac{\pi}{2}}$$

So, for a special case the two bodies of equal mass, make right angle between their directions after the collision.